

# Factorization of nonlinear supersymmetry in one-dimensional Quantum Mechanics. I: general classification of reducibility and analysis of the third-order algebra

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*We study possible factorizations of supersymmetric (SUSY) transformations in the one-dimensional quantum mechanics into chains of elementary Darboux transformations with nonsingular coefficients. A classification of irreducible (almost) isospectral transformations and of related SUSY algebras is presented. The detailed analysis of SUSY algebras and isospectral operators is performed for the third-order case.*

## 1. Introduction: definitions and notation of the SUSY QM

The concept of supersymmetric Quantum Mechanics (SUSY QM) represents an algebraic form of transformations of (complete or partial) spectral equivalence between different dynamical systems [1]–[5]. At present, there is a number of reviews [6]–[12] devoted to development and various applications of the SUSY QM; the reader is referred to these reviews, which are addressed for a more detailed study of this approach to construction of isospectral systems. Isospectral transformations of that kind are the Darboux-Moutard-Crum transformations [13], [14]–[17], which are known in the theory of ordinary differential equations for a long time <sup>1</sup>. In the simplest cases, intertwining of two differential operators (for instance, Hamiltonians of one-dimensional quantum systems) by means of Darboux operators entails their factorization into differential multipliers which are formed by the same Darboux operators (Schrödinger factorization [18], [19] and its generalizations [20], [21]). However, in general, this is not the case, and both interrelation between pairs of dynamical operators ("Hamiltonians") with (almost <sup>2</sup>) equivalent spectra and structure of operators which generate the spectral equivalence are not that simple [22] – [25]. Precisely this interrelation in the one-dimensional QM is the focus of the present paper. In particular, we present rigorously justified answers to the following questions: in what cases can the higher-order Darboux-Crum transformations be constructed with the help of a sequence of intertwining transformations of lower order which relate a chain of (almost) isospectral intermediate Hamiltonians with real nonsingular <sup>3</sup> potentials; what are elementary blocks for a nonsingular factorization of intertwining operators; in what way is the irreducibility of elementary blocks of isospectral transformations indicated in the SUSY algebra and in the structure of kernels of those transformations? The structure of the paper is as follows. After a short reminder of notation and basic definitions of SUSY theory of isospectral transformations we formulate basic theorems on the structure of a polynomial SUSY algebra and on minimization of this algebra up to its essential part (proofs of these theorems can be found in our preceding paper [26]). Then we present a classification of irreducible (almost) isospectral transformations and related SUSY algebras (partially described in [12], [27]–[31]). Next, we define a potential class  $K$  that is invariant under transformations of the Darboux-Crum type and formulate two theorems on reducibility of differential operators of spectral equivalence transformations. The paper is completed with a detailed analysis of the third-order SUSY algebras and isospectral operators as a first stage in proving above-mentioned theorems on reducibility. A complete proof will be published in a forthcoming issue.

Let us start with a definition of the SUSY algebra and notation of its components. Consider two one-dimensional Hamiltonians of the Schrödinger type  $h^+ = -\partial^2 + V_1(x)$  and  $h^- = -\partial^2 + V_2(x)$ ,  $\partial \equiv d/dx$ , which

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Translated from *Zapiski Nauchnykh Seminarov POMI*, Vol. 335, 2006, pp.22-49 .

<sup>1</sup>In the monograph [17] the Darboux transformations are given for a wider class of partial differential equations including non-stationary Schrödinger one and some nonlinear equations.

<sup>2</sup>We say that operators have almost equivalent spectra if their spectra are different only at a finite number of eigenvalues.

<sup>3</sup>In this case, the potentials are sufficiently smooth, but potentials having singularities weaker than  $1/x^2$  are also acceptable.

are defined on the entire axis and have nonsingular potentials  $V_{1,2}(x)$ . We assemble the Hamiltonians into a super-Hamiltonian,

$$H = \begin{pmatrix} h^+ & 0 \\ 0 & h^- \end{pmatrix}. \quad (1)$$

Assume that the Hamiltonians  $h^+$  and  $h^-$  have an (almost) equal energy spectrum of bound states and equal spectral densities of the continuous spectrum part; let such an equivalence be provided by the Darboux-Crum [13, 14] operators  $q_N^\pm$  with the help of intertwining,

$$h^+ q_N^+ = q_N^+ h^-, \quad q_N^- h^+ = h^- q_N^-. \quad (2)$$

Further on, we restrict ourselves to differential Darboux-Crum operators of finite order  $N$ ,

$$q_N^\pm = \sum_{k=0}^N w_k^\pm(x) \partial^k, \quad w_N^\pm \equiv (\mp 1)^N, \quad (3)$$

with real, sufficiently smooth coefficients  $w_k^\pm(x)$ . In this case, in the fermion number representation, the nonlinear  $\mathcal{N} = 1$  SUSY QM is formed by means of nilpotent supercharges,

$$Q_N = \begin{pmatrix} 0 & q_N^+ \\ 0 & 0 \end{pmatrix}, \quad \bar{Q}_N = \begin{pmatrix} 0 & 0 \\ q_N^- & 0 \end{pmatrix}, \quad Q_N^2 = \bar{Q}_N^2 = 0. \quad (4)$$

Obviously, the intertwining relations (2) lead to the supersymmetry of the Hamiltonian  $H$ ,

$$[H, Q_N] = [H, \bar{Q}_N] = 0. \quad (5)$$

This nonlinear SUSY algebra is closed by the following relation between the supercharges and Hamiltonian,

$$\{Q_N, \bar{Q}_N\} = P_N(H), \quad (6)$$

where  $P_N(H)$  is a differential operator of  $2N$ th order commuting with the Hamiltonian. Depending on a relation between the supercharges  $Q_N, \bar{Q}_N$  (the intertwining operators  $q_N^\pm$ ), the operator  $P_N(H)$  can be either a polynomial of the Hamiltonian if the intertwining operators are connected by the operation of transposition:  $q_N^+ = (q_N^-)^t \equiv \sum_{k=0}^N (-\partial)^k w_k^-(x)$ , or a function of both the Hamiltonian and a differential symmetry operator of odd order in derivatives (see a detailed analysis and references in [26]). In our paper, we confine ourselves with the first case in which the conjugated supercharge is produced by transposition,  $\bar{Q}_N = Q_N^t$  (a relevant theorem on the structure of such a SUSY is formulated below).

## 2. Basic theorems on the structure of QM with a nonlinear SUSY

**Theorem 1 (on supersymmetric algebra with transposition symmetry).**

Let  $\phi_n^\mp(x)$ ,  $n = 1, \dots, N$  be a basis in  $\ker q_N^\mp$ :

$$q_n^\mp \phi_n^\mp = 0, \quad q_N^- = (q_N^+)^t. \quad (7)$$

Then:

1) the action of the Hamiltonians  $h^\pm$  on the functions  $\phi_n^\pm(x)$  is described by constant  $N \times N$  matrices,

$$h^\pm \phi_n^\mp = \sum_{m=1}^N S_{nm}^\pm \phi_m^\mp \quad n = 1, \dots, N; \quad (8)$$

2) the closure of the supersymmetry algebra takes a polynomial form,

$$\{Q, Q^t\} = \det[E\mathbf{I} - \mathbf{S}^+]_{E=H} = \det[E\mathbf{I} - \mathbf{S}^-]_{E=H} \equiv P_N(H), \quad (9)$$

where  $\mathbf{I}$  is an identity matrix and  $\mathbf{S}^\pm$  is the matrix with entries  $S_{nm}^\pm$ .

**Corollary 1.** The spectra of the matrices  $\mathbf{S}^+$  and  $\mathbf{S}^-$  are equal.

In what follows, for an intertwining operator, its matrix  $\mathbf{S}$  is defined as the matrix which is related to operator in the same way as  $\mathbf{S}^\pm$  are related to  $q^\mp$ . In this case, we do not specify the basis in the kernel of the intertwining operator in which the matrix  $\mathbf{S}$  is chosen if we concern only with spectral characteristics of the matrix or, that is the same, spectral characteristics of the restriction of the corresponding Hamiltonian to the kernel of the intertwining operator considered (cf. (8)).

A basis in the kernel of the intertwining operator in which the matrix  $\mathbf{S}$  of this operator has a Jordan form is called *canonical*; elements of a canonical basis are called *transformation functions*.

Assume that the intertwining operators  $q_N^\pm$  are represented as a product of the intertwining operators  $k_{N-M}^\pm$  and  $p_M^\pm$ ,  $0 < M < N$  so that

$$\begin{aligned} q_N^+ &= p_M^+ k_{N-M}^+, & q_N^- &= k_{N-M}^- p_M^-; & p_M^+ h_M &= h^+ p_M^+, & p_M^- h^+ &= h_M p_M^-; \\ k_{N-M}^+ h^- &= h_M k_{N-M}^+, & k_{N-M}^- h_M &= h^- k_{N-M}^-, & h_M &= -\partial^2 + v_M(x), \end{aligned} \quad (10)$$

where the coefficients  $k_{N-M}^\pm$  and  $p_M^\pm$  as well as the potential  $v_M(x)$  may be complex and/or singular. The Hamiltonian  $h_M$  is called *intermediate* with respect to  $h^+$  and  $h^-$ . In this case, by Theorem 1, the spectrum of the matrix  $\mathbf{S}$  of the operator  $q_N^\pm$  is a union of the spectra of the matrices  $\mathbf{S}$  for the operators  $k_{N-M}^\pm$  and  $p_M^\pm$ .

The potentials  $V_1(x)$  and  $V_2(x)$  of the Hamiltonians  $h^+$  and  $h^-$  are interrelated by the equation

$$V_2(x) = V_1(x) - 2[\ln W(x)]'', \quad (11)$$

where  $W(x)$  is the Wronskian of elements of an arbitrary (a canonical as well) basis in  $\ker q_N^-$ . The validity of Eq. (11) follows from the Liouville-Ostrogradsky relation and the equality of coefficients at  $\partial^N$  in  $q_N^- h^+$  and  $h^- q_N^-$  (see the intertwining in (2)).

An intertwining operator  $q_N^\pm$  is called *minimizable* if this operator can be presented in the form

$$q_N^\pm = P(h^\pm) p_M^\pm = p_M^\pm P(h^\mp), \quad (12)$$

where  $p_M^\pm$  is an operator of order  $M$  which intertwines the same Hamiltonians as  $q_N^\pm$  (i.e.  $p_M^\pm h^\mp = h^\pm p_M^\pm$ ) and  $P(h^\pm)$  is a polynomial of degree  $(N - M)/2 > 0$ . Otherwise the intertwining operator  $q_N^\pm$  is named as *non-minimizable*.

The following theorem contains necessary and sufficient conditions under which an intertwining operator is minimizable or not (a proof can be found in [26]).

**Theorem 2 (on minimization of an intertwining operator)**

An intertwining operator  $q_N^\pm$  can be presented in the form

$$q_N^\pm = p_M^\pm \prod_{l=1}^m (\lambda_l - h^\mp)^{\delta k_l}, \quad (13)$$

where  $p_M^\pm$  is a nonminimizable operator intertwining the same Hamiltonians as  $q_N^\pm$  (so that  $p_M^\pm h^\mp = h^\pm p_M^\pm$ ), if and only if a Jordan form of the matrix  $\mathbf{S}$  of the operator  $q_N^\pm$  has  $m$  pairs (and no more) of Jordan cells with equal eigenvalues  $\lambda_l$  such that, for the  $l$ -th pair,  $\delta k_l$  is the order of the smallest cell and  $k_l + \delta k_l$  is the order of the largest cell. In this case,  $M = N - 2 \sum_{l=1}^m \delta k_l = \sum_{l=1}^n k_l$ , where the  $k_l$ ,  $m + 1 \leq l \leq n$  are orders of the remaining unpaired Jordan cells.

**Remark 1.** A Jordan form of the matrix  $\mathbf{S}$  of the intertwining operator  $q_N^\pm$  cannot have more than two cells with the same eigenvalue  $\lambda$ ; otherwise  $\ker(\lambda - h^\mp)$  includes more than two linearly independent elements.

**Corollary 2.** Jordan forms of the matrices  $\mathbf{S}$  of the operators  $q_N^+$  and  $q_N^-$  coincide up to permutation of Jordan cells.

If a Jordan form of the matrix  $\mathbf{S}$  of an intertwining operator has cells of order higher than one, then the corresponding canonical bases contains not only formal solutions of the Schrödinger equation but also formal associated functions, which are defined as follows [32].

A function  $\psi_{n,i}(x)$  is called a *formal associated function of  $i$ -th order* of the Hamiltonian  $h$  for a spectral value  $\lambda_n$  if

$$(h - \lambda_n)^{i+1} \psi_{n,i} \equiv 0, \quad \text{and} \quad (h - \lambda_n)^i \psi_{n,i} \not\equiv 0. \quad (14)$$

The term 'formal' emphasizes that this function is not necessarily normalizable (not necessarily belongs to  $L_2(\mathbb{R})$ ). In particular, an associated function  $\psi_{n,0}$  of zero order is a formal eigenfunction of  $h$  (not necessarily a normalizable solution of the homogeneous Schrödinger equation).

### 3. Classification of really (ir)reducible SUSY transformations

The intertwining operator  $q_N^\pm$  is called (*really*)*reducible* if this operator can be presented as a product of two nonsingular intertwining operators (with real coefficients)  $k_{N-M}^\pm$  and  $p_M^\pm$ ,  $0 < M < N$  so that Eqs. (10) are valid and the intermediate Hamiltonian  $h_M$  has a real nonsingular potential. Otherwise  $q_N^\pm$  is called (*really*)*irreducible*.

Really irreducible, nonminimizable, intertwining operators of second order with real coefficients can be divided into three types [12].

A *really irreducible intertwining operator of I type* is a differential intertwining operator with real coefficients for which eigenvalues of the matrix  $\mathbf{S}$  have nontrivial imaginary parts and are mutually complex conjugate.

Let us show that any intertwining operator  $q_2^-$  satisfying this definition is, in fact, really irreducible (the case of  $q_2^+$  is treated similarly). Indeed, let  $\varphi_{1,2}^-(x)$  be a canonical basis of  $\ker q_2^-$  such that  $h^+ \varphi_{1,2}^- = \lambda_{1,2} \varphi_{1,2}^-$ ,  $\lambda_1^* = \lambda_2 \neq \lambda_1$ . Assume that  $q_2^-$  is reducible, *i.e.*, there exist intertwining operators  $k_1^-$  and  $p_1^-$  with real nonsingular coefficients such that

$$q_2^- = k_1^- p_1^-, \quad p_1^- h^+ = h_1 p_1^-, \quad k_1^- h_1 = h^- k_1^-, \quad (15)$$

where  $h_1$  is an intermediate Hamiltonian with a real nonsingular potential. Obviously, a basis in the kernel of  $p_1^-$  consists either of  $\varphi_1^-$  or of  $\varphi_2^-$ . We restrict ourselves to the case of  $\varphi_1^-$  since the case of  $\varphi_2^-$  can be considered in the same manner. Then  $p_1^- = \partial - (\varphi_1^-)' / \varphi_1^-$ , and, consequently,  $f(x) = (\varphi_1^-)' / \varphi_1^-$  is a real-valued function. But then

$$\varphi_1^-(x) = C e^{\int f(x) dx}, \quad C = \text{Const};$$

hence,

$$V_1(x) - \lambda_1 = \frac{(\varphi_1^-)''(x)}{\varphi_1^-(x)} = f^2(x) + f'(x)$$

is a real-valued function as well, and we get a contradiction with the condition that  $\lambda_1^* \neq \lambda_1$ . Thus, any operator that satisfies the above definition is indeed really irreducible.

The degenerate case  $V_{2,1}(x) = \text{Const}$  should be singled out. In this case,  $h^+ = h^- = h_1$ , and the canonical basis  $\ker q_2^\pm$  can be chosen in the form

$$\varphi_1^\pm(x) = e^{kx}, \quad \varphi_2^\pm(x) = e^{k^*x}, \quad k \neq k^* \quad (16)$$

so that eigenvalues of the matrix  $\mathbf{S}$  of the operator  $q_2^\pm$  and the operator itself are as follows:

$$h^\mp \varphi_{1,2}^\pm = \lambda_{1,2}^\mp \varphi_{1,2}^\pm, \quad \lambda_1^\mp = V_{2,1} - k^2, \quad \lambda_2^\mp = V_{2,1} - k^{*2}, \quad q_2^\pm = \partial^2 - 2\text{Re } k \partial + |k|^2. \quad (17)$$

Note that potentials of the intermediate Hamiltonians which correspond to two possible factorizations of a really irreducible intertwining operator  $q_2^\pm$  of the I type into intertwining operators of first order, *i.e.*,

$$V_{2,1}(x) - 2[\ln \varphi_{1,2}^\pm(x)]'', \quad (18)$$

where  $\varphi_{1,2}^+(x)$  ( $\varphi_{1,2}^-(x)$ ) is a canonical basis in  $\ker q_2^+$  ( $\ker q_2^-$ ), always have a nontrivial imaginary part (see [23]) with the only exception of the case  $V_{2,1}(x) = \text{Const}$ .

A *really irreducible intertwining operator of the II type* is a differential intertwining operator  $q_2^\pm$  of second order with real coefficients such that:

- (1) eigenvalues of the matrix  $\mathbf{S}$  of the operator  $q_2^\pm$  are real and different;
- (2) both elements  $\varphi_1^\pm(x)$  and  $\varphi_2^\pm(x)$  of a canonical basis of  $\ker q_2^\pm$  have zeroes.

The irreducibility of intertwining operators satisfying this definition follows from the fact that otherwise the equalities,

$$q_2^+ = p_1^+ k_1^+, \quad p_1^+ h_1 = h^+ p_1^+, \quad k_1^+ h^- = h_1 k_1^+ \quad (19)$$

take place, or, according to (15), a basis in  $\ker k_1^+$  ( $\ker p_1^+$ ) consists either of  $\varphi_1^+(x)$  or of  $\varphi_2^+(x)$ , and the potential of the intermediate Hamiltonian  $h_1$  is described by one of Eqs. (18), *i.e.*, has a singularity(ies) by the second item of the definition. We also note that potentials of intermediate Hamiltonians which correspond to two possible singular factorizations of a really irreducible intertwining operator of the II type into intertwining operators of first order given by (18) are real since the  $\varphi_{1,2}^\pm(x)$  can be always chosen real.

A really irreducible intertwining operator of the III type is a differential intertwining operator  $q_2^\pm$  of second order with real coefficients such that:

- (1) the eigenvalues  $\lambda_{1,2}$  of the matrix  $\mathbf{S}$  of the operator  $q_2^\pm$  are equal,  $\lambda_1 = \lambda_2$ ;
- (2) a canonical basis in  $\ker q_2^\pm$  consists of formal eigenfunctions,  $\varphi_{10}^\pm(x)$ , and associated functions,  $\varphi_{11}^\pm(x)$ , of the Hamiltonian  $h^\mp$  which assemble into a Jordan cell,

$$h^\mp \varphi_{10}^\pm = \lambda_1 \varphi_{10}^\pm, \quad (h^\mp - \lambda_1) \varphi_{11}^\pm = \varphi_{10}^\pm;$$

- (3)  $\varphi_{10}^\pm(x)$  has at least one root.

The irreducibility of an intertwining operator satisfying this definition follows from the fact that otherwise equalities (19) take place, or, according to (15), a basis in  $\ker k_1^+$  ( $\ker p_1^-$ ) consists of  $\varphi_{10}^\pm(x)$ , and a potential of the intermediate Hamiltonian  $h_1$  is described by the equation

$$V_{2,1}(x) - 2[\ln \varphi_{10}^\pm(x)]'', \quad (20)$$

i.e., has a singularity(ies) by the third item of the definition. The potential of the intermediate Hamiltonian, which corresponds to the only possible singular factorization of a really irreducible intertwining operator of the III type into intertwining operators of first order given by (20), is real since the  $\varphi_{10}^\pm(x)$  can always be chosen real.

Obviously, other types of really irreducible nonminimizable intertwining operators of second order do not exist.

Further on, we formulate two assertions which characterize reducibility of intertwining operators of any order in an exhaustive way:

assertion (1) of Theorem 3 on the reducibility of a nonminimizable intertwining operator with real spectrum of the matrix  $\mathbf{S}$ , multiplied by an appropriate polynomial of the Hamiltonian, into (a product of) intertwining operators of first order;

assertion (2) of Theorem 4 on the reducibility of a nonminimizable intertwining operator with arbitrary spectrum of the matrix  $\mathbf{S}$  into (a product of) intertwining operators of first order and irreducible second-order intertwining operators of the I, II and III type.

## 4. Theorems on complete reducibility of intertwining operators

In what follows, we use a class  $K$  of potentials  $V(x)$  such that:

- 1)  $V(x)$  is a real-valued function from  $C_{\mathbb{R}}^\infty$ ;
- 2) there exist numbers  $R_0 > 0$  and  $\varepsilon > 0$  ( $R_0$  and  $\varepsilon$  depend on  $V(x)$ ) such that the inequality  $V(x) \geq \varepsilon$  takes place for any  $|x| \geq R_0$ ;
- 3) the functions

$$\left( \int_{\pm R_0}^x \sqrt{|V(x_1)|} dx_1 \right)^2 \left( \frac{|V'(x)|^2}{|V(x)|^3} + \frac{|V''(x)|}{|V(x)|^2} \right) \quad (21)$$

are bounded for  $x \geq R_0$  and  $x \leq -R_0$ , respectively.

In addition, we discuss normalizability and nonnormalizability of functions at  $+\infty$  and/or at  $-\infty$ ; these properties are defined as follows.

A function  $f(x)$  is called *normalizable at  $+\infty$  (at  $-\infty$ )* if there exists a real number  $a_+$  ( $a_-$ ) such that

$$\int_{a_+}^{+\infty} |f(x)|^2 dx < +\infty \quad \left( \int_{-\infty}^{a_-} |f(x)|^2 dx < +\infty \right). \quad (22)$$

Otherwise  $f(x)$  is called *nonnormalizable at  $+\infty$  (at  $-\infty$ )*.

**Theorem 3. (on reducibility of "dressed" nonminimizable intertwining operators)**  
Assume that the following conditions are satisfied:

1)  $h^+ = -\partial^2 + V_1(x)$ ,  $V_1(x) \in K$ , and the potential  $V_2(x)$  of the Hamiltonian  $h^-$  is real and continuous;  
 2)  $h^+$  and  $h^-$  are intertwined by a nonminimizable differential operator of  $N$ th order  $q_N^-$  with coefficients from  $C_{\mathbb{R}}^2$ , so that

$$q_N^- h^+ = h^- q_N^-; \quad (23)$$

3) the algebraic multiplicity of  $\lambda_i$ , the  $i$ th eigenvalue of the matrix  $\mathbf{S}$  for the operator  $q_N^-$ , is equal to  $k_i$ ,  $i = 1, \dots, n$ , so that  $k_1 + \dots + k_n = N$ ; all of the numbers  $\lambda_i$  are real and satisfy the inequalities

$$0 \geq \lambda_1 > \lambda_2 > \dots > \lambda_n; \quad (24)$$

4)  $\Lambda$  is the spectrum of the matrix  $\mathbf{S}$  of the operator  $q_N^-$ ;  
 5)  $E_{i\pm}$ ,  $i = 0, 1, 2, \dots$ , is the energy of the  $i$ th (from below) bound state of  $h^\pm$ ;  $N^\pm$  is the number of bound states of  $h^\pm$  with energies of which are included into  $\Lambda$ ;  $N_\pm$  be a number of bound states of  $h^\pm$  with energies not exceeding  $\lambda_1$ ;

6)

$$P_\pm(E) = \prod_{E_{i\pm} < \lambda_1, E_{i\pm} \notin \Lambda} (E - E_{i\pm}). \quad (25)$$

Then: 1)  $V_2(x) \in K$ ; coefficients of  $q_N^-$  belong to  $C_{\mathbb{R}}^\infty$  and are real;  $q_N^+ = (q_N^-)^t$  has real coefficients from  $C_{\mathbb{R}}^\infty$  and intertwines  $h^+$  and  $h^-$ , so that

$$h^+ q_N^+ = q_N^+ h^-; \quad (26)$$

2)  $P_+(E) \equiv P_-(E)$ ; the degree of  $P_\pm(E)$  is equal to  $N_+ - N^+ = N_- - N^-$ ;  
 3) the operator  $q_N^\mp P_\pm(h^\pm)$  intertwines  $h^+$  and  $h^-$  and can be presented as a product of  $N+N_++N_- - N^+ - N^-$  intertwining operators of first order with real coefficients from  $C_{\mathbb{R}}^\infty$ , so that:  
 a) potentials of all the intermediate Hamiltonians belong to  $K$ ;  
 b) the eigenvalue of the matrix  $\mathbf{S}$  of the  $l$ -th operator (from the right) in the factorization under consideration is equal to  $E_{l-1,\pm}$ ,  $l = 1, \dots, N_\pm$  and an element of the kernel of this operator is normalizable at both infinities;  
 c) the eigenvalue of the matrix  $\mathbf{S}$  of the  $l$ -th operator (from the left) in the factorization under consideration is equal to  $E_{l-1,\mp}$ ,  $l = 1, \dots, N_\mp$  and an element of the kernel of this operator is nonnormalizable at both infinities;  
 d) the set of eigenvalues of the matrices  $\mathbf{S}$  for operators from the  $N_\pm + 1$ -th to the  $N_\pm + N - N^+ - N^-$ -th one (from the right) in the factorization under consideration coincides with<sup>4</sup>  $\Lambda \setminus (\{E_{i+}\} \cup \{E_{i-}\})$ . In addition, the eigenvalue of the matrix  $\mathbf{S}$  for an operator of this group does not decrease as the number of the operator increases (from the right to left); a basis element of the kernel of any operator in this group is normalizable at one of the infinities only.

**Theorem 4. (on complete reducibility of nonminimizable intertwining operators)**

Assume that the following conditions are satisfied:

1)  $h^+ = -\partial^2 + V_1(x)$ ,  $V_1(x) \in K$ ; the potential  $V_2(x)$  of the Hamiltonian  $h^-$  is real and continuous;  
 2)  $h^+$  and  $h^-$  are intertwined by a nonminimizable differential operator  $q_N^-$  of  $N$ th order with real coefficients from  $C_{\mathbb{R}}^2$ , so that

$$q_N^- h^+ = h^- q_N^-; \quad (27)$$

3) the algebraic multiplicity of  $\lambda_i$ , the  $i$ th eigenvalue of matrix  $\mathbf{S}$  for operator  $q_N^-$ , is equal  $k_i$ ,  $i = 1, \dots, n$ , so that  $k_1 + \dots + k_n = N$ ; the set of values  $\lambda_i$  contains  $M$  real values and  $L$  pairs of mutually complex conjugate ones, so that  $M + 2L = n$ ; the numbers  $i = 1, \dots, M$  correspond to real  $\lambda_i$ , and  $\lambda_i > \lambda_{i+1}$ ,  $i = 1, \dots, M - 1$ ;

4) if  $\lambda_1$  is real, then  $\lambda_1 \leq 0$ ;

5)  $E_{i\pm}$ ,  $i = 0, 1, 2, \dots$  is the energy of the  $i$ th bound state (from below) of  $h^\pm$ ;  $K_\pm = \max\{i : \lambda_i > E_{0\pm}\}$ , if  $\lambda_1 > E_{0\pm}$ , and  $K_\pm = 0$ , if either  $\lambda_1 \leq E_{0\pm}$  or  $\text{Im } \lambda_1 \neq 0$ .

Then: 1)  $V_2(x) \in K$ ; coefficients of  $q_N^-$  belong to  $C_{\mathbb{R}}^\infty$ ;  $q_N^+ = (q_N^-)^t$  has real coefficients from  $C_{\mathbb{R}}^\infty$  and intertwines  $h^+$  and  $h^-$ , so that

$$h^+ q_N^+ = q_N^+ h^-; \quad (28)$$

<sup>4</sup>In this formula, one has to take into account multiplicities of eigenvalues as follows: if  $\lambda$  is contained in  $\Lambda$  with algebraic multiplicity  $K_1$ , in  $\{E_{i+}\}$  with multiplicity  $K_2$  and in  $\{E_{i-}\}$  with multiplicity  $K_3$  (obviously,  $K_2$  and  $K_3$  can take values 0 and 1 only), then the value  $\lambda$  is contained in  $\Lambda \setminus (\{E_{i+}\} \cup \{E_{i-}\})$  with multiplicity  $K_1 - K_2 - K_3$  if  $K_1 > K_2 + K_3$  or is not contained if  $K_1 \leq K_2 + K_3$ .

2)  $q_N^\mp$  can be presented as a product of really irreducible intertwining operators of first and second order with real coefficients from  $C_{\mathbb{R}}^\infty$ , so that:

- a) potentials of all the intermediate Hamiltonians belong to  $K$ ;
- b) the first

$$J_1 = \sum_{i=M+1}^{M+L} k_i \quad (29)$$

operators from the right in the factorization of  $q_N^\mp$  under consideration have an order 2 and are really irreducible operators of the I type ; in addition, one can realize that the related pairs of mutually complex conjugated eigenvalues of the matrix  $\mathbf{S}$  operator  $q_N^\mp$  are ordered arbitrarily;

- c) the second (from the right) group of operators in the factorization under consideration consists of

$$J_{2\mp} = N - 2J_1 - 2J_{3\mp}, \quad (30)$$

operators of first order, where

$$J_{3\mp} = \left[ \frac{1}{2} \sum_{i=1}^{K_\mp} k_i \right], \quad (31)$$

and

(i) if  $\sum_{i=1}^{K_\mp} k_i$  is even, then the eigenvalue of the matrix  $\mathbf{S}$  for the operator  $q_N^\mp$  which corresponds to the  $l$ th (from the right) of these operators does not exceed the eigenvalue related to the  $l+1$ -th operator,  $l = 1, \dots, J_{2\mp} - 1$ ;

(ii) if  $\sum_{i=1}^{K_\mp} k_i$  is odd, then the eigenvalue of the matrix  $\mathbf{S}$  for the operator  $q_N^\mp$  which corresponds to the  $l$ th (from the right) of these operators does not exceed the eigenvalue related to the  $(l+1)$ th operator,  $l = 1, \dots, J_{2\mp} - 2$ ;  $\lambda_{K_\mp}$  is an eigenvalue of the  $(J_{2\mp} - 1)$ th operator and  $\lambda_{K_\mp+1}$  is an eigenvalue of the  $(J_{2\mp})$ th operator; in this case, the latter eigenvalue is equal to  $E_{0\mp}$ ;

d) the third (from the right) and the last group of operators in the factorization under consideration consists of  $J_{3\mp}$  really irreducible operators of II and III type, wherein the largest of eigenvalues of the matrix  $\mathbf{S}$  for the operator  $q_N^\mp$  which corresponds to the  $l$ th of these operators (from the right) does not exceed the smallest eigenvalue of the matrix  $\mathbf{S}$  for the operator  $q_N^\mp$  which corresponds to the  $(l+1)$ th of these operators,  $l = 1, \dots, J_{3\mp} - 1$ .

**Remark 1.** If  $E_{0\mp}$  is not an eigenvalue of the matrix  $\mathbf{S}$  for the operator  $q_N^\mp$ , then  $\sum_{i=1}^{K_\mp} k_i$  is even since otherwise the eigenvalue of  $q_N^\mp q_N^\pm \equiv \prod_{i=1}^n (h^\mp - \lambda_i)^{k_i}$  at the ground state wave function of  $h^\mp$  is negative.

Proofs of these theorems will be published in a forthcoming issue.

The next aim of this paper is to show rigorously that any nonminimizable intertwining operator of third order with real coefficients is really reducible<sup>5</sup>. Such a proof is a necessary stage in the study of reducibility of intertwining operators of arbitrary order. For this purpose, we first derive differential equations for Wronskians of subsets of a canonical basis of the kernel of the intertwining operator  $q_N^\mp$  of an arbitrary order  $N$ . These equations form a base of the proof of reducibility for an arbitrary intertwining operator of third order (Theorem 5) and also can be used, for instance, to examine reducibility of intertwining operator in the general case where a Jordan form of its matrix  $\mathbf{S}$  is a single Jordan cell.

## 5. Derivation of system of equations for partial Wronskians

Let  $\phi_j(x)$ ,  $j = 1, \dots, N$  be a canonical basis in  $\ker q_N^\mp$  and let  $\lambda_j$ ,  $j = 1, \dots, N$  be an eigenvalue of the matrix  $\mathbf{S}$  for the operator  $q_N^\mp$  corresponding to the Jordan cell to which  $\phi_j(x)$  is related. It is shown in [26], Lemma 1, that:

<sup>5</sup>For preliminary consideration of this issue, see in [33, 34]

1) the intertwining operator  $q_N^-$  can be presented as follows

$$q_N^- = r_1^- \dots r_N^-, \quad (32)$$

where the Darboux operators

$$r_j^- = \partial + \chi_j(x), \quad j = 1, \dots, N, \quad (33)$$

can be chosen to satisfy the equalities

$$r_j^- \dots r_N^- \phi_j = 0, \quad j = 1, \dots, N; \quad (34)$$

2) the following relations take place

$$\begin{aligned} (r_j^-)^t r_j^- + \lambda_j &= r_{j+1}^- (r_{j+1}^-)^t + \lambda_{j+1} \equiv h_j, \quad j = 1, \dots, N-1, \\ (r_N^-)^t r_N^- + \lambda_N &= h^+ \equiv h_N, \quad r_1^- (r_1^-)^t + \lambda_1 = h^- \equiv h_0; \end{aligned} \quad (35)$$

3) the intermediate Hamiltonians  $h_j$ ,  $j = 1, \dots, N-1$  have the Schrödinger form:

$$h_j = -\partial^2 + v_j(x), \quad v_j(x) = \chi_j^2(x) - \chi_j'(x) + \lambda_j = \chi_{j+1}^2(x) + \chi_{j+1}'(x) + \lambda_{j+1}, \quad (36)$$

$$V_1(x) \equiv v_N(x) = \chi_N^2(x) - \chi_N'(x) + \lambda_N, \quad V_2(x) \equiv v_0(x) = \chi_1^2(x) + \chi_1'(x) + \lambda_1, \quad (37)$$

but in general with complex and/or singular potentials;

4) the intertwining relations

$$h_l r_{l+1}^- = r_{l+1}^- h_{l+1}, \quad (r_{l+1}^-)^t h_l = h_{l+1} (r_{l+1}^-)^t, \quad l = 0, \dots, N-1. \quad (38)$$

are realized.

Let us introduce generalized Crum determinants

$$W_j(x) = \begin{vmatrix} \phi_N(x) & \phi_N'(x) & \dots & \phi_N^{(N-j)}(x) \\ \phi_{N-1}(x) & \phi_{N-1}'(x) & \dots & \phi_{N-1}^{(N-j)}(x) \\ \dots & \dots & \dots & \dots \\ \phi_j(x) & \phi_j'(x) & \dots & \phi_j^{(N-j)}(x) \end{vmatrix}, \quad j = 1, \dots, N. \quad (39)$$

By (34), the following expressions for intertwining operators  $r_j^- \dots r_N^-$  are valid:

$$r_j^- \dots r_N^- = \frac{1}{W_j(x)} \begin{vmatrix} \phi_N(x) & \phi_N'(x) & \dots & \phi_N^{(N-j+1)}(x) \\ \dots & \dots & \dots & \dots \\ \phi_j(x) & \phi_j'(x) & \dots & \phi_j^{(N-j+1)}(x) \\ 1 & \partial & \dots & \partial^{N-j+1} \end{vmatrix}, \quad j = 1, \dots, N \quad (40)$$

therefore,

$$r_j^- \dots r_N^- \phi_{j-1} = \frac{W_{j-1}(x)}{W_j(x)}, \quad j = 2, \dots, N. \quad (41)$$

Furthermore, the intermediate superpotentials  $\chi_j(x)$  are as follows:

$$\chi_j(x) = -\frac{[W_j(x)/W_{j+1}(x)]'}{W_j(x)/W_{j+1}(x)} = -\frac{W_j'(x)}{W_j(x)} + \frac{W_{j+1}'(x)}{W_{j+1}(x)}, \quad j = 1, \dots, N, \quad W_{N+1}(x) \equiv 1. \quad (42)$$

To obtain differential equations satisfied by the Wronskians  $W_j(x)$ , we convert the expression

$$q_j^- (q_j^-)^t \frac{W_{j-1}(x)}{W_j(x)}, \quad j = 2, \dots, N, \quad (43)$$

in two different ways. On the one hand, we take into account (35), (41) and intertwining (38) to show that

$$q_j^- (q_j^-)^t \frac{W_{j-1}(x)}{W_j(x)} = (h_{j-1} - \lambda_j) q_j^- \dots q_N^- \phi_{j-1} = q_j^- \dots q_N^- (h^+ - \lambda_j) \phi_{j-1} = (\lambda_{j-1} - \lambda_j) \frac{W_{j-1}(x)}{W_j(x)}, \quad (44)$$



Obviously, equalities (44) are valid not only if  $\phi_{j-1}$  is a formal eigenfunction of  $h^+$  but also if  $\phi_{j-1}$  is a formal associated function of  $h^+$ . On the other hand, Eq. (43) can be transformed as follows,

$$\begin{aligned} q_j^- \left[ -\partial - \frac{W_j'}{W_j} + \frac{W_{j+1}'}{W_{j+1}} \right] \frac{W_{j-1}(x)}{W_j(x)} &= q_j^- \left[ -\frac{W_{j-1}'}{W_j} + \frac{W_{j-1}W_j'}{W_j^2} - \frac{W_j'W_{j-1}}{W_j^2} + \frac{W_{j+1}'W_{j-1}}{W_{j+1}W_j} \right] \\ &= - \left[ \partial - \frac{W_j'}{W_j} + \frac{W_{j+1}'}{W_{j+1}} \right] \left[ \frac{W_{j+1}}{W_j} \left( \frac{W_{j-1}}{W_{j+1}} \right)' \right] = -2 \left( \frac{W_{j+1}}{W_j} \right)' \left( \frac{W_{j-1}}{W_{j+1}} \right)' - \frac{W_{j+1}}{W_j} \left( \frac{W_{j-1}}{W_{j+1}} \right)'' \\ &= -\frac{W_j}{W_{j+1}} \left[ \left( \frac{W_{j+1}}{W_j} \right)^2 \left( \frac{W_{j-1}}{W_{j+1}} \right)' \right]', \end{aligned} \quad (45)$$

where we use (42). Finally we obtain the equations,

$$\left[ \left( \frac{W_{j+1}}{W_j} \right)^2 \left( \frac{W_{j-1}}{W_{j+1}} \right)' \right]' + (\lambda_{j-1} - \lambda_j) \left( \frac{W_{j+1}}{W_j} \right)^2 \frac{W_{j-1}}{W_{j+1}} = 0, \quad j = 2, \dots, N. \quad (46)$$

For further purposes, it is convenient to introduce the functions  $w_j = W_j'/W_j$ ,

$$w_j' - w_{j+2}' + w_j^2 - w_{j+2}^2 - 2w_{j+1}(w_j - w_{j+2}) + \lambda_j - \lambda_{j+1} = 0, \quad j = 1, \dots, N-1, \quad (47)$$

in system (46). In addition, supplementing system (47) with the equation

$$w_N' + w_N^2 + \lambda_N - V_1 = 0 \quad (48)$$

(i.e., the Schrödinger equation for  $W_N$  rewritten for  $w_N$ ) and summing up the last  $N - n + 1$  equations of the new system, we get the relations

$$w_n' + w_{n+1}' + (w_n - w_{n+1})^2 + \lambda_n - V_1 = 0, \quad n = 1, \dots, N. \quad (49)$$

## 6. Parametric formulas for partial Wronskians

If  $N = 3$  system (47), (48) takes the following form

$$w_1' - w_3' + w_1^2 - w_3^2 - 2w_2(w_1 - w_3) + \lambda_1 - \lambda_2 = 0, \quad (50)$$

$$w_2' + w_2^2 - 2w_3w_2 + \lambda_2 - \lambda_3 = 0, \quad (51)$$

$$w_3' + w_3^2 - V_1 + \lambda_3 = 0. \quad (52)$$

Let us introduce the function

$$G(x) = \frac{1}{2} [w_1'(x) + w_1^2(x) - V_1(x) + \lambda_1 + \lambda_2 + \lambda_3]. \quad (53)$$

Equalities (50) and (52) imply the identity

$$w_2(w_1 - w_3) = G - \lambda_2. \quad (54)$$

In order to derive a formula which expresses  $w_3$  in terms of  $G$ , let us compare two expressions for  $w_2'$ : the expression, obtained by differentiation of the equality

$$w_2 = \frac{G - \lambda_2}{w_1 - w_3}, \quad (55)$$

which follows from (54), and the expression deduced from (51) after a substitution of (55) into (51) instead of  $w_2$ . By solving the appearing quadratic equation for  $w_3$ , we come to the equality

$$w_3 = w_1 + \frac{G' - \sqrt{(G')^2 + 4P_3(G)}}{2(G - \lambda_3)}, \quad P_3(\lambda) \equiv (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) \quad (56)$$

for a certain branch of  $\sqrt{(G')^2 + 4P_3(G)}$ . It follows from (54) and (56) that

$$w_2 = \frac{G' + \sqrt{(G')^2 + 4P_3(G)}}{2(G - \lambda_1)}. \quad (57)$$

Note that formulas (55), (56) and (57) are valid if  $G(x)$  is different from an identical constant that equal one of the numbers  $\lambda_j$ . Below, we show that the latter condition always takes place for an intertwining operator  $q_3^-$  that cannot be stripped-off.

Now we derive formulas which express  $w_1$  in terms of  $G$ . We substitute (56) into (50) instead of  $w_3$  and obtain  $w_1$  from the resulting expression to deduce that if

$$[G'(x)]^2 + 4P_3(G(x)) \neq 0 \quad (58)$$

then the equality

$$w_1 = \frac{G'' + 2P_3'(G)}{2\sqrt{(G')^2 + 4P_3(G)}} \quad (59)$$

holds; if for some interval

$$[G'(x)]^2 + 4P_3(G(x)) \equiv 0, \quad G'(x) \neq 0, \quad (60)$$

on some interval, then

$$w_1(x) \equiv 0 \quad (61)$$

on this interval, and the potentials  $V_1(x)$  and  $V_2(x)$  are identical by (11).

## 7. Smoothness of potentials and coefficients of intertwining operators

The following lemma indicates how smooth are the  $V_2(x)$  and coefficients of the intertwining operators  $q_3^\pm$  for a given smoothness of  $V_1(x)$ .

**Lemma 1.** Assume that: 1)  $h^\pm = -\partial^2 + V_{1,2}(x)$ ,  $V_1(x) \in C_{\mathbb{R}}^n$ ,  $n \geq 3$ , and  $V_2(x) \in C_{\mathbb{R}}$ ;

2)

$$q_3^- = \partial^3 + \alpha(x)\partial^2 + \beta(x)\partial + \gamma(x); \quad \alpha(x), \beta(x), \gamma(x) \in C_{\mathbb{R}}^2; \quad (62)$$

3)  $q_3^-$  intertwines  $h^+$  and  $h^-$ , so that

$$q_3^- h^+ = h^- q_3^-. \quad (63)$$

Then:

1)  $\alpha(x) \in C_{\mathbb{R}}^{n+1}$ ,  $\beta(x) \in C_{\mathbb{R}}^n$ ,  $\gamma(x) \in C_{\mathbb{R}}^{n-1}$ , and  $V_2(x) \in C_{\mathbb{R}}^n$ ;

2) the operators  $q_3^-$  and  $q_3^+ = (q_3^-)^t$  can be presented in the form

$$q_3^\mp = \pm \partial^3 + g_2(x)\partial^2 + [g_2'(x) \mp 2g_1(x)]\partial + [g_0(x) \mp g_1'(x)], \quad (64)$$

where

$$g_2(x) = \alpha(x), \quad g_1(x) = [\alpha'(x) - \beta(x)]/2, \quad g_0(x) = \gamma(x) + [\alpha''(x) - \beta'(x)]/2;$$

in addition  $g_2(x) \in C_{\mathbb{R}}^{n+1}$ ,  $g_1(x) \in C_{\mathbb{R}}^{n+1}$ ,  $g_0(x) \in C_{\mathbb{R}}^{n-1}$ ;

3)  $q_3^+$  intertwines  $h^+$  and  $h^-$ , so that

$$h^+ q_3^+ = q_3^+ h^-. \quad (65)$$

*Proof.* Let us check first that

$$\alpha(x) \in C_{\mathbb{R}}^{n-1}, \quad \beta(x) \in C_{\mathbb{R}}^{n-1}, \quad \gamma(x) \in C_{\mathbb{R}}^{n-1}. \quad (66)$$

Indeed, inclusions (66) follow from relations (32) and (40) for  $j = 1$  and  $N = 3$ , from the fact that  $\phi_1(x)$ ,  $\phi_2(x)$  and  $\phi_3(x)$  belong to  $C_{\mathbb{R}}^{n+2}$  as formal eigenfunctions and (possibly) associated functions of  $h^+$ , and from the fact that  $W_1(x)$ , a Wronskian of basis elements in  $\ker q_3^-$ , does not have zeroes.

From the intertwining condition (63), we derive the following system of equations:

$$V_2(x) - V_1(x) = 2\alpha'(x), \quad (67)$$

$$\alpha(x)[V_2(x) - V_1(x)] - 3V_1'(x) = \alpha''(x) + 2\beta'(x), \quad (68)$$

$$\beta(x)[V_2(x) - V_1(x)] - 2\alpha(x)V_1'(x) - 3V_1''(x) = \beta''(x) + 2\gamma'(x), \quad (69)$$

$$\gamma(x)[V_2(x) - V_1(x)] - \beta(x)V_1'(x) - \alpha(x)V_1''(x) - V_1'''(x) = \gamma''(x). \quad (70)$$

Relations (66), (67) and condition 1 imply that  $V_2(x) \in C_{\mathbb{R}}^{n-2}$ . We deduce from (66), (68), condition 1, and the inclusion  $V_2(x) \in C_{\mathbb{R}}^{n-2}$  that  $\alpha''(x) \in C_{\mathbb{R}}^{n-2}$ , i.e.,  $\alpha(x) \in C_{\mathbb{R}}^n$ . From (67), condition 1 and the fact that  $\alpha(x) \in C_{\mathbb{R}}^n$  it follows that  $V_2(x) \in C_{\mathbb{R}}^{n-1}$ . It follows from (66), (69), condition 1 and the inclusions  $V_2(x) \in C_{\mathbb{R}}^{n-1}$  and  $\alpha(x) \in C_{\mathbb{R}}^n$  that  $\beta(x) \in C_{\mathbb{R}}^n$ . From (68), condition 1, and the inclusions  $V_2(x) \in C_{\mathbb{R}}^{n-1}$ , and  $\alpha(x)$  and  $\beta(x) \in C_{\mathbb{R}}^n$  it follows that  $\alpha(x) \in C_{\mathbb{R}}^{n+1}$ . Finally, it follows from (67), condition 1 and the inclusion  $\alpha(x) \in C_{\mathbb{R}}^{n+1}$  that  $V_2(x) \in C_{\mathbb{R}}^n$ . Thus, the first statement is proved.

The validity of equality (64) is easily verified with the help of straightforward calculations. The fact that  $g_2(x)$  and  $g_0(x)$  are in  $C_{\mathbb{R}}^{n+1}$  and  $C_{\mathbb{R}}^{n-1}$ , respectively, and equality (65) are obvious. Finally, to show that  $g_1(x)$  belongs to  $C_{\mathbb{R}}^{n+1}$  we refer to the equalities

$$3V_1(x) + \alpha'(x) + 2\beta(x) \in C_{\mathbb{R}}^{n+1}, \quad (71)$$

$$3V_1'(x) + \beta'(x) + 2\gamma(x) \in C_{\mathbb{R}}^n, \quad (72)$$

and

$$V_1'(x) + \gamma(x) \in C_{\mathbb{R}}^n, \quad (73)$$

which, in turn, follow from equalities (68)–(70) since  $V_{1,2}(x) \in C_{\mathbb{R}}^n$ ,  $\alpha(x) \in C_{\mathbb{R}}^{n+1}$ ,  $\beta(x) \in C_{\mathbb{R}}^n$  and  $\gamma(x) \in C_{\mathbb{R}}^{n-1}$ . Lemma 1 is proved.

**Corollary 3.** By calculating the coefficient at  $\partial^2$  in  $q_3^-$  with the help of (32), (33) and (42), we deduce that

$$\alpha(x) \equiv g_2(x) \equiv -\frac{W_1'(x)}{W_1(x)} \equiv -w_1(x). \quad (74)$$

Hence, under the conditions of Lemma 1

$$w_1(x) \in C_{\mathbb{R}}^{n+1}, \quad \text{and} \quad W_1(x) = Ce^{\int w_1(x) dx} \in C_{\mathbb{R}}^{n+2}. \quad (75)$$

## 8. Parametric formulas for coefficients of intertwining operators

It was shown in [34] that the potentials  $V_1(x)$ ,  $V_2(x)$  and coefficients of the intertwining operator  $q_3^+$  can be parameterized by a single function which was denoted  $W(x)$  in [34]. It is not difficult to check that this function is connected with  $G(x)$  by the relation

$$W = G - \frac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3). \quad (76)$$

For convenience, we give parametric formulas obtained in [34] in the notation of the present work,

$$V_{1,2} = g_2^2 \mp g_2' - 2G + \lambda_1 + \lambda_2 + \lambda_3, \quad (77)$$

$$g_1 = \frac{1}{2}[g_2^2 - 3G + \lambda_1 + \lambda_2 + \lambda_3], \quad (78)$$

$$g_0 = g_2'' - g_2[g_2^2 - 3G + \lambda_1 + \lambda_2 + \lambda_3] + \frac{1}{2}\sqrt{(G')^2 + 4P_3(G)}. \quad (79)$$

Let us emphasize that, in contrast to (59), parameterizations (77)–(79) are valid for any case. Parameterization (59) which supplements (77)–(79), (see, in addition, (74)) is derived in [34] as well, but only under the following conditions

$$G(x) \neq \text{Const}, \quad [G'(x)]^2 + 4P_3(G(x)) \neq 0, \quad (80)$$

whereas in the present work we derive this parameterization under a weaker condition (58). The case of conditions (60), (61) was not considered in [34].

## 9. Relations between parameterization function and partial Wronskians

In the following lemma, we indicate basic relations between the parameterization function  $G(x)$  and Wronskians of a part of canonical basis elements in the kernel of  $q_3^-$ .

**Lemma 2.** Assume that: 1) the conditions of Lemma 1 are fulfilled; 2)  $W_{jk} = \phi'_j \phi_k - \phi_j \phi'_k$ ; 3)  $\sqrt{(G')^2 + 4P_3(G)}$  is the same branch of the root as above. Then:

1) if a Jordan form of the matrix  $\mathbf{S}$  for the operator  $q_3^-$  contains three Jordan cells of first order,

$$h^+ \phi_1 = \lambda_1 \phi_1, \quad h^+ \phi_2 = \lambda_2 \phi_2, \quad h^+ \phi_3 = \lambda_3 \phi_3, \quad (81)$$

then the following identities hold:

$$W'_{kl} = (\lambda_l - \lambda_k) \phi_k \phi_l, \quad \left( \frac{\phi_j}{W_1} \right)' = \varepsilon_{jkl} (\lambda_k - \lambda_l) \frac{W_{jk} W_{jl}}{W_1^2}, \quad (82)$$

$$G - \lambda_j = \varepsilon_{jkl} (\lambda_j - \lambda_k) (\lambda_j - \lambda_l) \frac{\phi_j W_{kl}}{W_1}, \quad (83)$$

$$G' + \sqrt{(G')^2 + 4P_3(G)} = 2(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1) \frac{\phi_1 \phi_2 \phi_3}{W_1}, \quad (84)$$

$$G' - \sqrt{(G')^2 + 4P_3(G)} = 2(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1) \frac{W_{12} W_{23} W_{31}}{W_1^2}, \quad (85)$$

where  $j, k, l$  is an arbitrary permutation of 1, 2, 3 and summation for a repeated index is not performed; in addition, the branch of  $\sqrt{(G')^2 + 4P_3(G)}$  is independent of the choice of numbering of canonical basis elements;

2) if a Jordan form of the matrix  $\mathbf{S}$  for the operator  $q_3^-$  contains a Jordan cell of first order and a Jordan cell of second order:

$$h^+ \phi_1 = \lambda_1 \phi_1, \quad h^+ \phi_2 = \lambda_2 \phi_2 + \phi_3, \quad h^+ \phi_3 = \lambda_3 \phi_3, \quad \lambda_3 = \lambda_2, \quad (86)$$

then the following identities hold:

$$W'_{13} = (\lambda_2 - \lambda_1) \phi_1 \phi_3, \quad W'_{23} = -\phi_3^2, \quad \left( \frac{\phi_1}{W_1} \right)' = \frac{W_{13}^2}{W_1^2}, \quad \left( \frac{\phi_3}{W_1} \right)' = (\lambda_1 - \lambda_2) \frac{W_{13} W_{23}}{W_1^2}, \quad (87)$$

$$G - \lambda_1 = (\lambda_1 - \lambda_2)^2 \frac{\phi_1 W_{23}}{W_1}, \quad G - \lambda_2 = (\lambda_1 - \lambda_2) \frac{\phi_3 W_{13}}{W_1}, \quad (88)$$

$$G' + \sqrt{(G')^2 + 4P_3(G)} = -2(\lambda_1 - \lambda_2)^2 \frac{\phi_1 \phi_3^2}{W_1}, \quad G' - \sqrt{(G')^2 + 4P_3(G)} = 2(\lambda_1 - \lambda_2)^2 \frac{W_{13}^2 W_{23}}{W_1^2}, \quad (89)$$

and the branch of  $\sqrt{(G')^2 + 4P_3(G)}$  is independent of the choice of numbering of canonical basis elements;

3) if the Jordan cell of the matrix  $\mathbf{S}$  for the operator  $q_3^-$  consists of a single Jordan cell of third order:

$$h^+ \phi_1 = \lambda_1 \phi_1 + \phi_2, \quad h^+ \phi_2 = \lambda_2 \phi_2 + \phi_3, \quad h^+ \phi_3 = \lambda_3 \phi_3, \quad \lambda_3 = \lambda_2 = \lambda_1, \quad (90)$$

then the following identities hold:

$$W'_2 = -\phi_3^2, \quad \left( \frac{\phi_3}{W_1} \right)' = \frac{W_2^2}{W_1^2}, \quad (91)$$

$$G - \lambda_1 = \frac{\phi_3 W_2}{W_1}, \quad (92)$$

$$G' + \sqrt{(G')^2 + 4P_3(G)} = -2 \frac{\phi_3^3}{W_1}, \quad G' - \sqrt{(G')^2 + 4P_3(G)} = 2 \frac{W_2^3}{W_1^2}. \quad (93)$$

**Proof.** Identities (82), (87) and (91) are easily checked with the help of straightforward calculations in which we use relations (81), (86) and (90).

Identities (83), (88) and (92) follow from identity (54) - for convenience, we write the latter identity in the case considered in the form

$$G - \lambda_2 = -\frac{W'_{23}}{W_{23}} \frac{(\phi_3/W_1)'}{\phi_3/W_1}; \quad (94)$$

as well, those identities follow from identities (82), (87), (91) and from the fact that we can renumber elements of canonical basis so that any given eigenvalue of the matrix  $\mathbf{S}$  for the operator  $q_3^-$  gets index 2 (see (94)).

In the case  $G \not\equiv \lambda_j$ ,  $j = 1, 2, 3$  identities (84), (85), (89), and (93) follow from identities (83), (88) and (92) and relations (56), (57). Before we prove identities (84), (85), (89) and (93) in the case  $G \equiv \lambda_j$ ,  $j = 1, 2, 3$  let us show that this case is equivalent to the existence of two Jordan cells with the same eigenvalue  $\lambda_j$  in a Jordan form of the matrix  $\mathbf{S}$  for the operator  $q_3^-$ . In the latter case, the validity of the desired identities is obvious.

Identities (82), (87), (91) and the fact, that any formal eigenfunction of the Hamiltonian (different from identical zero) can have zeroes of first order only imply that the right-hand sides of expressions (83), (88) and (92) either are identical zeroes (if there are two Jordan cells with the same eigenvalue in a Jordan form of the matrix  $\mathbf{S}$  for the operator  $q_3^-$ ) or can have zeroes of the order not exceeding four. Thus, the identity  $G \equiv \lambda_j$  holds on some interval if and only if this identity holds on whole axis which is equivalent to the existence of two Jordan cells with the same eigenvalue  $\lambda_j$  in a Jordan form of the matrix  $\mathbf{S}$  for the operator  $q_3^-$ . Hence, identities (84), (85), (89) and (93) are proved.

To show that the branch of  $\sqrt{(G')^2 + 4P_3(G)}$  is independent of the numbering of canonical basis elements, one can renumber these elements, derive formulas similar to (84), (85), (89) and (93) for new numbering and compare the results. Lemma 2 is proved.

**Corollary 4.** In the proof of Lemma 2, it was shown that for any  $j$  the function  $G(x) - \lambda_j$  either is identical zero on the whole axis or can have zeroes only of order not exceeding four. In addition, the relation  $G(x) \equiv \lambda_j$  is equivalent to the existence of two Jordan cells with the same eigenvalue  $\lambda_j$  in a Jordan form of the matrix  $\mathbf{S}$  for the operator  $q_3^-$ , which in view of Theorem 2, is equivalent to the possibility to strip-off the operators  $q_3^\mp$ .

**Corollary 5.** Under the conditions of Lemma 1, the functions  $\phi_1(x)$ ,  $\phi_2(x)$  and  $\phi_3(x)$  belong to  $C_{\mathbb{R}}^{n+2}$ ; hence it follows from identities (82), (87) and (91), that the Wronskians  $W_{kl}(z)$  (case 1),  $W_{13}(x)$  and  $W_{23}(x)$  (case 2), and  $W_2(x)$  (case 3) belong to  $C_{\mathbb{R}}^{n+3}$ . We apply these inclusions, take differences of identities (84) and (85); (89); (93), refer to inclusions (75), and take into account that  $W_1(x)$  has no zeroes (as the Wronskian of a basis in  $\ker q_3^-$ ) to show that

$$\sqrt{(G')^2 + 4P_3(G)} \in C_{\mathbb{R}}^{n+2}. \quad (95)$$

In its turn, inclusion (95) and formulas (84), (89) and (93) provide that

$$G(x) \in C_{\mathbb{R}}^{n+3}. \quad (96)$$

**Corollary 6.** If coefficients of  $q_3^-$  are real, then coefficients of the polynomial  $P_3(h^\pm) = q_3^\pm q_3^\mp$  are real as well. Hence, either all of the numbers  $\lambda_j$  are real or one of these numbers is real and two are mutually complex conjugate. Without loss of generality, we assume that elements of the canonical basis in  $\ker q_3^-$  that correspond to real  $\lambda_j$ , are chosen real, and elements that correspond to complex conjugate  $\lambda_j$  are complex conjugate. Then, if all of  $\lambda_j$  are real, the root  $\sqrt{(G')^2 + 4P_3(G)}$  is real for any  $x \in \mathbb{R}$  in view of (84), (89) and (93). If there is pair of complex conjugate values  $\lambda_j$  (obviously, this is possible only if all of  $\lambda_j$  are different), then  $W_1(x)$  is purely imaginary (since complex conjugation of  $W_1(x)$  corresponds to a permutation of two lines in the definition of  $W_1(x)$ ) and  $\sqrt{(G')^2 + 4P_3(G)}$  is also real for any  $x \in \mathbb{R}$  by virtue of (84). Thus,

$$[G'(x)]^2 + 4P_3(G(x)) \geq 0, \quad x \in \mathbb{R}. \quad (97)$$

## 10. Lower bound of the parameterization function

A lower bound for the parameterization function  $G(x)$  is given by the following lemma.

**Lemma 3.** *If, under the conditions of Lemma 1, the intertwining operator  $q_3^-$  cannot be stripped-off, coefficients of  $q_3^-$  are real and  $\lambda_3$  is the minimal real eigenvalue of the matrix  $\mathbf{S}$  for the operator  $q_3^-$ , then the inequality*

$$G(x) > \lambda_3, \quad x \in \mathbb{R}, \quad (98)$$

holds. **Proof.** First we show that the inequality

$$G(x) \geq \lambda_3, \quad x \in \mathbb{R} \quad (99)$$

takes place. Assume that for some point  $x_0 \in \mathbb{R}$ , inequality (99) is violated. Then  $P_3(G(x_0)) < 0$  and, consequently, by (97), the derivative  $G'(x_0) \neq 0$ . Moreover,  $G'(x)$  does not vanish on the entire interval which contains  $x_0$  and on which the inequality  $G(x) < \lambda_3$  holds. Hence,  $G(x)$  either strictly increases or strictly decreases on this interval. Obviously the interval is not bounded from the left (right), if  $G(x)$  increases (decreases) on it. Let us show that the assumption about the violation of (99) leads to a contradiction. For definiteness, we consider the case where  $G(x)$  increases on the above-mentioned interval. The case of decreasing  $G(x)$  is treated similarly. By inequality (97), the inequality  $G'(x)/\sqrt{-P_3(G(x))} \geq 2$  is valid for any point of the considered interval. Integrating the latter inequality from  $x$  to  $x_0$ , we deduce that

$$\int_{G(x)}^{G(x_0)} \frac{dG}{\sqrt{-P_3(G)}} \geq 2(x_0 - x), \quad x < x_0. \quad (100)$$

The left-hand side of inequality (100) is bounded for  $x \rightarrow -\infty$  while its right-hand side tends to  $+\infty$ . This contradiction proves inequality (99).

To prove that  $G(x) - \lambda_3$  has no zeroes, we use identity (59) which expresses  $g_2(x)$  in terms of  $G(x)$  (see also (74)). Let us assume that there is a point  $x_0 \in \mathbb{R}$  such that  $G(x_0) = \lambda_3$ . Since  $q_3^-$  cannot be stripped-off, Corollary 4 and inequality (99) at the point  $x_0$  imply that the function  $G(x) - \lambda_3$  has a zero of even order  $2n$ ,  $G'(x)$  has a zero of order  $2n - 1$ , and  $G''(x)$  has a zero of order  $2n - 2$ , where  $n$  is either 1 or 2; in addition, it is obvious that

$$G''(x_0) \geq 0. \quad (101)$$

First we consider the case where  $\lambda_3$  is a zero of  $P_3(\lambda)$  of order one. In this case, inequality (99) and the fact that the order of the root of  $G(x) - \lambda_3$  is even imply condition (58), which allows us to use formula (59). Finally, since

$$P'_3(G(x_0)) = (\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2) > 0, \quad (102)$$

and inequality (101) holds, the right-hand side of (59) at the point  $x_0$  is infinite, which contradicts to (75). Hence,  $G(x)$  cannot equal  $\lambda_3$ .

Now we assume that  $\lambda_3$  is a zero of  $P_3(\lambda)$  of order two or three. In this case, the numerator of (59) has, obviously, a zero of order  $2n - 2$  at the point  $x_0$  and the denominator has a zero of order  $2n - 1$ . Hence,  $g_2(x)$  has a pole at the point  $x_0$ , which is impossible. Thus,  $G(x)$  cannot equal  $\lambda_3$ , and Lemma 3 is proved.

## 11. Theorem on reducibility of intertwining operators of the third order

The assertion that any intertwining operator of the third order with real coefficients is really reducible is described by the following theorem.

**Theorem 5.** Assume that the conditions of Lemma 1 are satisfied and that the intertwining operator  $q_3^-$  cannot be stripped-off has real coefficients. Let  $\lambda_3$  be the minimal real eigenvalue of the matrix  $\mathbf{S}$  for the operator  $q_3^-$ . Then there exist intertwining operators  $p_1^\pm$  and  $k_1^\pm$  of the first order and  $p_2^\pm$  and  $k_2^\pm$  of the second order such that:

- 1) coefficients of  $p_1^\pm$  and  $k_1^\pm$  are real, and, in addition, coefficients of these operators at  $\partial^0$  belong to  $C_{\mathbb{R}}^{n+1}$ ;
- 2) coefficients of  $p_2^\pm$  and  $k_2^\pm$  are real, and, in addition, coefficients of these operators at  $\partial$  and  $\partial^0$  belong to  $C_{\mathbb{R}}^{n+2}$  and  $C_{\mathbb{R}}^n$ , respectively;

3)

$$p_1^+ = (p_1^-)^t, \quad k_1^+ = (k_1^-)^t, \quad p_2^+ = (p_2^-)^t, \quad k_2^+ = (k_2^-)^t; \quad (103)$$

- 4) the matrices  $\mathbf{S}$  for the operators  $p_1^\pm$  and  $k_1^\pm$  consist of  $\lambda_3$ ;

5)

$$q_3^- = k_2^- p_1^- = k_1^- p_2^-, \quad q_3^+ = p_1^+ k_2^+ = p_2^+ k_1^+, \quad (104)$$

$$p_1^- h^+ = h_1 p_1^-, \quad k_2^- h_1 = h^- k_2^-, \quad p_2^- h^+ = h_2 p_2^-, \quad k_1^- h_2 = h^- k_1^-, \quad (105)$$

$$h^+ p_1^+ = p_1^+ h_1, \quad h_1 k_2^+ = k_2^+ h^-, \quad h^+ p_2^+ = p_2^+ h_2, \quad h_2 k_1^+ = k_1^+ h^-, \quad (106)$$

where  $h_1$  and  $h_2$  are intermediate Hamiltonians with real potentials from  $C_{\mathbb{R}}^n$ .

**Proof.** We consider the case with  $p_1^-$  and  $k_2^-$  only since the statements of Theorem 5 for the case of  $p_1^+$  and  $k_2^+$  are easily verifiable with the help of transposition, and the statement for the cases of  $k_1^{\pm}$  and  $p_2^{\pm}$  follows from the symmetry between  $h^+$  and  $h^-$ .

Let us define  $p_1^-$  and  $k_2^-$  by the equalities

$$p_1^- = r_3^-, \quad k_2^- = r_1^- r_2^-. \quad (107)$$

Then existence of an intermediate Hamiltonian  $h_1$  and intertwining (105) follows from relations (35) and (38). The fact that the potential of the Hamiltonian  $h_1$  given by the formula

$$V_1(x) - 2[\ln \phi_3(x)]'' \equiv V_1(x) - 2w_3'(x) \quad (108)$$

(see (11)) is real and belongs to the space  $C_{\mathbb{R}}^n$  follows from (56), (74), (97), (98), from the fact that  $g_2(x)$  is real, and from the inclusions (95), (96), and  $g_2(x) \in C_{\mathbb{R}}^{n+1}$  (see Lemma 1). In addition, the function

$$\chi_3(x) \equiv -\frac{\phi_3'}{\phi_3} \equiv -w_3, \quad (109)$$

which is the coefficient at  $\partial^0$  of the operator  $p_1^-$ , is obviously real and belongs to  $C_{\mathbb{R}}^{n+1}$ . To prove that coefficients of  $k_2^-$  are real and belong to the spaces of smooth functions of Theorem 5 we first apply relations (33), (42), (51), (52) and (54) to transform  $k_2^-$  to the form

$$k_2^- = \partial^2 + (w_3 - w_1)\partial + (G + V_1 - w_3^2 - w_1 w_3 - 2\lambda_3), \quad (110)$$

and then take into account the following statements:  $V_1(x)$ ,  $w_1(x) \equiv -g_2(x)$ ,  $\lambda_3$  (see the Theorem 5 conditions),  $G(x)$  and  $w_3(x)$  are real, identity (56) and inclusions (75), (95), and (96) hold,  $w_3(x) \in C_{\mathbb{R}}^{n+1}$ , and  $V_1(x) \in C_{\mathbb{R}}^n$ . Finally, the fact that the matrix  $\mathbf{S}$  for the operator  $p_1^-$  consists of  $\lambda_3$  follows from (35). Theorem 5 is proved.

The work was supported by the RFBR Grant 06-01-00186-a. The first author was supported by the Programs “Development of scientific potential of higher school”, grant RPN 2.1.1.1112 and “Leading scientific schools of Russia”, grant LSS 2.1.1.1112.

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